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U-Duality in supersymmetric Born-Infeld theory

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Abstract: The Born-Infeld theory of a toroidal D3-brane has an $SL(5, \mathbb{Z})$ U-duality symmetry. We investigate how this symmetry is reflected in the supersymmetry algebra. We propose an action of the group on the gauge theory fields in the BPS sector by introducing an extra field together with an additional symmetry, and argue for the U-invariance of the degeneracies of the BPS spectrum.

Keywords: Duality in Gauge Field Theories, D-branes, String Duality, Supersymmetry and Duality.

Contents

1. Introduction

The BPS-sector of toroidally compactified type II string theories is well known to be symmetric under a large discrete symmetry group, U-duality [1]. In fact, the half BPS particles (in the noncompact dimensions), arising from perturbative as well as non-perturbative ten-dimensional states, transform in a single multiplet of the U-duality group. Thus one can relate states such as wrapped D-branes, which are non-perturbative from the string point of view, to states that have a perturbative string description. In particular, U-duality predicts the degeneracies of string states and dual D-brane states to be equal. This prediction was checked for particular configurations by Vafa and Sen [2], who used the gauge theory description of D-branes [3].

The gauge theory on a D-brane provides a description of not only the free brane itself, but also of bound states with lower dimensional branes and strings. These 1/4 BPS states, preserving 8 of the 32 supersymmetries, are identified with configurations with non-zero electric and magnetic fluxes [4]. For low enough dimensions, ($d \leq 4+1$) the flux quantum numbers of the gauge theories are sufficient to describe all bound states with lower dimensional objects than the brane itself, so one may wonder how much of the U-duality remains valid in the field theory limit. This question was addressed in [5, 6], partly motivated by the study of matrix theory [7]. Although the Yang-Mills hamiltonian is not U-invariant, it was found in [6] that, remarkably, the part of the BPS-spectrum containing single BPS-states (as opposed to those consisting of multiple states) exhibits U-invariant degeneracies.

The gauge theory on the brane arises as an effective theory describing the dynamics of the open strings glued to the D-brane. To first approximation this theory is indeed a Yang-Mills theory. However, when one wants to incorporate fields with large field strength compared to the string scale, higher derivative terms begin to play a role. It was shown by Leigh [8] that the α' expansion sums to a Born-Infeld gauge theory. (The extension of this theory to non-abelian gauge groups is still not fully understood; for proposals and discussion see [9, 10]). It turns out that the U-symmetry that was not fully realised by the Yang-Mills theory is more apparent in the Born-Infeld theory. In [11] the BPS masses of the compactified D4-brane BI-theory were computed, and shown to be consistent with the full $\mathrm{SO}(5,5,\mathbb{Z})$ symmetry. In particular, the rank of the group, i.e. the number of D4-branes, appears on an equal footing with the other quantum numbers.

In this paper we will investigate the action of U-duality on the supersymmetric version of the Born-Infeld theory (as presented in [12]), specialised to four dimensions. The intention is to generalise the method and results of [6], where the four-dimensional Yang-Mills theory was investigated, to the Born-Infeld theory. The expectation is that, due to the more symmetric form of BI theory compared to Yang-Mills theory, we may find better agreement with the U-duality (in this case $\mathrm{SL}(5,\mathbb{Z})$) predictions from string theory (in particular, we hope to also incorporate multiple BPS states in a U-dual degeneracy formula).

We derive the (U-invariant) superalgebra and the BPS-equations (which of course coincide with the result in the bosonic theory). We find that we can express the equations naturally in terms of a field in the antisymmetric representation of $\mathrm{SL}(5)$, whose zero modes are the quantum numbers. In identifying this field with the gauge fields we still have one extra local symmetry, which we may use to set the rank to a constant. In the context of classical solutions to the BPS equations, this allows us to indeed use U-duality as a solution generating technique.

In order to study the degeneracies of the quantum system, we follow the philosophy of [13, 6] in quantising the theory restricted to the space of supersymmetric configurations. In a simple case the quantisation is very similar to that in the Yang-Mills case, where the BPS sector was shown to be related to a matrix string theory. We propose an action of the U-duality on the quantum theory, where we interpret the extra local symmetry just mentioned as a conformal symmetry, and the fixing of the rank to a constant as an analogue of light cone gauge fixing. We discuss multiple BPS states and present a U-duality invariant degeneracy formula for arbitrary states, depending only on two invariants constructed from the quantum numbers.

Finally we make some speculative comments on the similarity of the structure of the four-dimensional Born-Infeld theory to the self-dual tensor theory living on an M5-brane.

2. SL(5)-invariance of the hamiltonian and the BPS masses

The type IIB three brane compactified on a torus can be described by the Born-Infeld gauge theory. Configurations of this gauge theory on a three torus can be labelled by various integer quantum numbers, namely electric and magnetic fluxes, and momenta around the torus. These gauge theory quantities reflect the various bound states that the three brane in the string theory can be in. Electric fluxes correspond to bound states with strings, while magnetic fluxes, via electro-magnetic duality, represent D-strings. The world brane momenta are of course just the momentum numbers of perturbative string theory on a torus. Finally, the rank of the gauge theory reflects the number of three branes. The Born-Infeld action is only known for the U(1) theory.

Configurations in string theory on a three torus enjoy a U-duality symmetry $SL(5,\mathbb{Z})$, which acts on the various quantum numbers. The aim of this paper is to investigate the extent to which this symmetry is reflected in the gauge theory on the brane. In this section we will first analyse the form of the hamiltonian of the (abelian) Born-Infeld theory, and demonstrate that the associated BPS mass, the minimal mass for a state with given quantum numbers, is compatible with the action of U-duality.

We begin with a derivation of the bosonic D3-brane hamiltonian along the lines of [11], and write it in a form that suggests an action of SL(5) on the various fields. For simplicity, we will set the various background tensor-fields to zero. The action is then

$$S_{BI} = \int d^3x\, dt\, \frac{-1}{g_s}\sqrt{\det(-G-F)}\,. \tag{2.1}$$

Here G is the metric on the toroidal three brane, in string units; g_s is the ten-dimensional string coupling.

The $SL(5,\mathbb{Z})$ will act on the zero modes of the following ten fields,

$$\begin{aligned} \text{electric fields}:\ E^i &= \frac{\delta\mathcal{L}}{\delta F_{0i}}\,, \\ \text{magnetic fields}:\ \epsilon_{ijk}B^k &= F_{ij} = \partial_{[i}A_{j]}\,, \\ \text{momenta}:\ P_i &= F_{ij}E^j\,, \\ \text{rank}:\ N &= 1\,. \end{aligned} \tag{2.2}$$

The zero-modes of these fields are the integer quantum numbers, and will be denoted by lower-case letters.

The hamiltonian squared can be derived to be, in terms of these fields,

$$\mathcal{H}^2 = \frac{\det G}{{g_s}^2}(N^2 + G^{ik}G^{jl}F_{ij}F_{kl}) + (E,P)\begin{pmatrix} G & 0 \\ 0 & G^{-1}\end{pmatrix}\begin{pmatrix} E \\ P\end{pmatrix}. \tag{2.3}$$

From the string theory point of view, the first line gives the contributions from D3 and D1-branes, while the last line contains the string winding and momentum modes.

To bring out the SL(5)-symmetry we organise the fields in an antisymmetric tensor,

$$M = \begin{pmatrix} 0 & P_3 & -P_2 & E^1 & F_{23} \\ -P_3 & 0 & P_1 & E^2 & F_{31} \\ P_2 & -P_1 & 0 & E^3 & F_{12} \\ -E^1 & -E^2 & -E^3 & 0 & 1 \\ -F_{23} & -F_{31} & -F_{12} & -1 & 0 \end{pmatrix} . \tag{2.4}$$

We also define the analogous matrix of zero-modes m_{ij}, with as entries the zero-modes of the fields.

The hamiltonian squared can be expressed as

$$\mathcal{H}^2 = -\frac{1}{2}\mathrm{Tr}\mathcal{G}M\mathcal{G}M \,, \tag{2.5}$$

with

$$\mathcal{G} = \begin{pmatrix} \frac{G}{\sqrt{\det G}} & 0 & 0 \\ 0 & \mathcal{G}_{44} & 0 \\ 0 & 0 & \mathcal{G}_{55} \end{pmatrix} , \tag{2.6}$$

and $\mathcal{G}_{44} = \sqrt{\det G}$, $\mathcal{G}_{55} = \sqrt{\det G}/{g_s}^2$. Non-zero off-diagonal entries would correspond to expectation values of the anti-symmetric tensor background fields.

SL(5) acts by conjugation on the fields M and the background $\mathcal{G}$, leaving $\mathcal{H}$ invariant. The SL(3) subgroup residing in the top-lefthand corner is of course just the structure group of the three-torus, leaving the coupling g_s fixed and acting on the momenta P_i, electric fields E^i and magnetic fields $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$ as vectors. The SL(2) subgroup acting on the 4, 5-indices is the familiar S-duality group, exchanging electric and magnetic fields and inverting the coupling. It also acts on $\det G$, when expressed in string units; as explained in [11], one should express the moduli in seven-dimensional Planck units to have a truly duality invariant formula. These two subgroups both leave the rank N fixed. The rest of SL(5) mixes the rank with the fields, and strictly speaking we only derived the hamiltonian for the case $N = 1$, for lack of knowledge of the non-abelian version of the Born-Infeld theory. However, based on the information we have from string theory, we should expect the SL(5) structure of the hamiltonian as given above to be correct. In the gauge theory, the transformations involving the rank should have an interpretation as Nahm transformations [6].

The matrix M_{ij} contains combinations of Born-Infeld fields that are not all independent. In particular, the momentum densities P_i are the components of the Poynting vector, the exterior product of the electric and magnetic fields. This seems to spoil the SL(5) symmetric structure. Remarkably, we can remedy this by viewing the entries of M as a priori independent fields, and in addition expressing the relations between them by imposing an SL(5) covariant vector K^i of constraints. We define this K using a five-dimensional ϵ symbol, defined via the metric $\mathcal{G}$, as

$$K^i = \frac{1}{8}\epsilon^{ijklm} M_{jk} M_{lm} \equiv *(M \wedge M)^i \,, \qquad (2.7)$$

so that in terms of the Born-Infeld fields

$$K = \Big(NP_i - (E \wedge B)_i, -P \cdot B, P \cdot E\Big) \,. \qquad (2.8)$$

For the case that $N = 1$, the first three components of the constraint $K = 0$ give the definition of the momenta P_i in terms of the electro-magnetic fields; the last two components are then automatically zero as well.

Using the constraint $K = 0$ we may derive the BPS mass formula for a given set of charges via the Bogomolny argument developed in [11]. We may write, for any unit five-vector $\hat{v}^i$,

$$\mathrm{Tr}\mathcal{G}M\mathcal{G}M = \frac{1}{2}\left(M_{ij} + \frac{1}{2}\epsilon_{ijklm}\hat{v}^k M^{lm}\right)\left(M^{ji} + \frac{1}{2}\epsilon^{jik'l'm'}\hat{v}_{k'} M_{l'm'}\right) \,, \qquad (2.9)$$

(where we use $\mathcal{G}$ to raise and lower indices), and split M_{ij} into one part containing only the zero modes m_{ij} and another with the fluctuations, M'_{ij}. From this we can, for any given $\hat{v}^i$, minimise the energy, and maximising this minimum over $\hat{v}^i$ we find the BPS bound

$$M^2_{BPS} = -\frac{1}{2}\mathrm{Tr}\mathcal{G}m\mathcal{G}m + 2\sqrt{k \cdot k} \,, \qquad (2.10)$$

where $k = *(m \wedge m) = (p_i - (e \wedge b)_i, -p \cdot b, p \cdot e)$ (recall that the lower case quantities are the zero modes of the corresponding fields). Note that the first term in (2.10) is precisely the expression for $\mathcal{H}^2$ with the zero modes of the fields inserted; we see therefore that for configurations having non-zero five-vector k^i (the 1/4 BPS states, as opposed to the 1/2 BPS states that have $k^i = 0$), the mass is strictly greater than the zero mode value. Hence, for such configurations the mass necessarily gets contributions from fluctuations in the fields.

This lower bound on the mass is only attained when the fields satisfy the BPS equations. These follow from the arguments leading to the BPS mass formula, and in this case require an expression involving the fluctuations in the fields to be proportional to the same expression with the zero modes:

$$(|k|M' + k * M') \sim (|k|m + k * m) \,. \qquad (2.11)$$

The proportionality factor is itself a function of space time. In the following we will want to interpret this rather peculiar BPS equation, and, in particular, since the rank N is part of the matrix M, to understand how to deal with possible fluctuations in N. In order to address these questions we will first rederive the above results from the supersymmetric Born-Infeld action, where the BPS equation follows from the requirement of unbroken symmetry. Then we will explore the solutions to the equations, using U-duality as a solution generating transformation. Here we will in particular demonstrate the consequences of transformations that formally give the rank N fluctuating contributions. Finally we will propose a method of quantisation of the theory restricted to its BPS space, motivated by the SL(5) symmetry, and demonstrate the U-invariance of the BPS degeneracies. First, however, we review the situation found by Hacquebord and Verlinde in their investigation of the theory in the Yang-Mills limit.

3. Review of the Yang-Mills case

The question of degeneracies of BPS states of the three-brane was first investigated in the context of $\mathcal{N}=4$ super Yang-Mills theory on the torus [6]. We will here first review this analysis, and then try to apply a similar method to the Born-Infeld case.

The U(N) super Yang-Mills theory is the reduction of ten-dimensional super Yang-Mills theory to four dimensions. Its bosonic fields are the vector potential, A_μ, and six scalar fields X^i. The theory has eight four-dimensional supersymmetries, four of which (Q_1) are the generators of the supersymmetries that are broken by the introduction of a brane, and act as shifts on the associated goldstone fermions λ. The other four generators (Q_2) are the unbroken supersymmetries, acting linearly.

In ten-dimensional notation, the two charges are given by

$$Q_1=\int \mathrm{Tr}\,\lambda, \quad Q_2=\int \mathrm{Tr}\,(E^i\Gamma_{0i}+\Gamma_{ij}F_{ij})\lambda\,. \tag{3.1}$$

Together they form a superalgebra

$$\{\overline{Q}_a,Q_b\}=\begin{pmatrix} N\Gamma_0 & 2e_i\Gamma_i-\Gamma_{0ij}f_{ij} \\ 2e_i\Gamma_i+\Gamma_{0ij}f_{ij} & 2\Gamma_0p^0+2\Gamma_ip^i\end{pmatrix}, \tag{3.2}$$

$(a,b=1,2)$ where lower case signifies the zero modes.

M(atrix) theory is the proposal to describe M-theory in its infinite momentum frame by means of the dynamics of zero-branes [7]. From this perspective, Yang-Mills theory on a three-torus is a description of M-theory on a three-torus. In [16] it was demonstrated that the Yang-Mills theory superalgebra is indeed equal to the M-theory algebra in the light cone limit, $N\to\infty$.

BPS states are states that are annihilated by certain combinations of the supercharges, depending on the quantum numbers. This implies that the variation of the fermions λ should vanish under such symmetries, from which one can derive equations that the bosonic fields should satisfy, the BPS equations [14]. These were determined in [6] to be

$$E'_i\kappa^i=0\,, \qquad E'_{[r}\kappa_{s]}=F'_{rs}\,. \tag{3.3}$$

The primes indicate that these equations only involve the non-zero modes of the fields; the vector κ is the three dimensional part of the SL(5) vector k (defined below equation (2.10)), representing the momentum minus its contribution from the zero modes of the fields:

$$\kappa = p - \frac{e \wedge b}{N}\,. \tag{3.4}$$

In a suitable gauge the BPS equations become equivalent to the free field equations of a chiral two-dimensional model, with no dependence on the directions orthogonal to the vector κ. If we choose coordinates such that the momentum points in the 1-direction,

$$\partial_0 A_i = \partial_1 A_i\,, \qquad \partial_0 X_I = \partial_1 X_I\,, \tag{3.5}$$

with $i = 2, 3$, $I = 1, \ldots, 6$. Furthermore, the U(N) valued fields are required to mutually commute, but they are allowed to be identified periodically along the spacelike circle modulo a permutation. From this it was argued in [6] that in the case of all electric and magnetic fluxes equal to zero, the solutions are characterised by the sigma model on

$$\frac{(\mathbb{R}^6 \times \mathbf{T}^2)^N}{S_N}\,, \tag{3.6}$$

where S_N is the Weyl group of U(N). The vector field is responsible for the $\mathbf{T}^2$, whereas the six scalars parametrise $\mathbb{R}^6$. The model therefore reduces to a matrix string theory [15]; its Hilbert space consists of all products of Hilbert spaces of strings of length n_i, such that their total length is N. In particular, [6] considered the sector involving one long string, where the S_N twists on the A, X fields are such that they are described by fields taking values on a circle whose length is precisely N times that of the physical circle. Quantising this system, it was found that the number of states of quantum numbers N, p_1 equaled $d(Np_1)$, defined by

$$\sum d(n) q^n = 256 \prod \left(\frac{1+q^n}{1-q^n}\right)^8\,. \tag{3.7}$$

The inclusion of electric or magnetic fluxes in the 1-direction was argued to effectively decrease the rank to $N' = \gcd(N, e_1, b_1)$. The reason is that in a U(N) gauge theory a magnetic flux b_1 is realised by having twisted boundary conditions for the SU(N) part of the gauge fields along the 2 and 3 directions. These twists should be two SU(N) transformations which commute up to a $\mathbb{Z}_N$ phase factor determined by the value of the magnetic flux modulo N. The point is that the S_N twists making the string in a long string should commute with these magnetic twists, in order not to introduce an additional magnetic flux in the 2 or 3 directions. This turns out to imply that the relevant permutation group can be maximally $S_{N'}$, thus effectively reducing the length of the single long string.

Therefore in this case the oscillation number of the long string adds up to $N' \times (p_1 - (e \wedge b)_1/N)$, which is precisely the greatest common divisor of the five-vector k. The resulting degeneracy is then precisely that of (single particle) string states, and in accordance with U-duality it only depends on the U-invariant quantity $|k| = \gcd(k)$.

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In the long string sector, Hacquebord and Verlinde therefore find a degeneracy of states in agreement with predictions by U-duality. Sectors consisting of several shorter strings, however, do not respect the symmetry; this is related to the fact that the Yang-Mills hamiltonian is not invariant under SL(5). We will see below that using the Born-Infeld theory, which does have a U-invariant hamiltonian, improves this situation.

4. Supersymmetric Born-Infeld theory

We now want to apply the same methods to study the BPS spectrum of the Born-Infeld theory. First we will use the supersymmetric generalisation of the theory, described in [12], to obtain expressions for the supercharges and their algebra. Remarkably, this algebra will turn out to be isomorphic to the superalgebra of M-theory compactified on a four-torus. This establishes that there exists an action of SL(5), which is the associated U-duality. Then we derive, from the requirement that the superalgebra should have eigenvalues equal to zero, the BPS mass found previously from the bosonic argument.

The supersymmetric Born-Infeld theory in ten dimensions was obtained in [12] in the simple form

$$S_{SBI} = -\int d^{10}x \sqrt{-\det\Big(\eta_{\mu\nu} + F_{\mu\nu} - 2\overline{\lambda}\Gamma_\mu \partial_\nu \lambda + \overline{\lambda}\Gamma^\rho \partial_\mu \lambda \overline{\lambda}\Gamma_\rho \partial_\nu \lambda\Big)}\,. \tag{4.1}$$

(The background has been further simplified to $g_s = 1, G = \eta$.) This result was derived from a Green-Schwarz type action with manifest target space covariance, by fixing the local κ-symmetry and going to static gauge. Half of the 32 supersymmetries of the target space are broken; these are realised non-linearly on the action (4.1). The fermions λ are the associated Goldstone fermions. The other sixteen supersymmetries act linearly. The transformations can be split into three parts: the simple supertranslations of the covariant theory, plus the complicated kappa-symmetry and coordinate transformations needed to restore the gauge. These add up to the rules [12]

$$\begin{aligned} \delta\overline{\lambda} &= \overline{\epsilon}_1 + \overline{\epsilon}_2\zeta + (\overline{\epsilon}_2\zeta - \overline{\epsilon}_1)\Gamma_\mu \lambda \partial_\mu \overline{\lambda} \\ \delta A_\mu &= (\overline{\epsilon}_2\zeta - \overline{\epsilon}_1)\Gamma_\mu\lambda + (\overline{\epsilon}_2\zeta - \overline{\epsilon}_1)\Gamma^\rho \lambda \partial_\rho A_\mu + \partial_\mu(\overline{\epsilon}_2\zeta - \overline{\epsilon}_1)\Gamma^\rho \lambda A_\rho\,. \end{aligned} \tag{4.2}$$

ζ is a particular expression involving wedge products of the field strength F contracted with gamma-matrices; its specific form in the case $d = 4$, putting all derivatives of the six scalars to zero, is given below. ϵ_1 corresponds to the broken supersymmetry, the unbroken one is associated to a combination of ϵ_1 and ϵ_2.

These expressions are valid in ten dimensions. The four-dimensional Born-Infeld theory we need can be easily obtained from this one by dimensional reduction.

Unfortunately, the action (4.1) and the associated transformation rules are those applying to a U(1) theory only. We will henceforth do computations in this abelian theory, although at some points the rank N will be written to bring out the symmetry more clearly. In the end we will make some remarks on the alterations that should occur when non-abelian theories are considered.

The programme now is to compute the supersymmetry algebra, and find bosonic configurations which are annihilated by half of the supersymmetries. To facilitate the computations, we will expand the action to quadratic order in the fermions λ; this is enough to find the purely bosonic central charges.

From the variations given in (4.2) it is straightforward to compute the two supercharge densities. We find, to linear order in the fermions,

$$J_1 = -2\Big(-P^0\Gamma_0\lambda + (E^i + P^i)\Gamma_i\lambda\Big)\,, \qquad J_2 = \zeta J_1\,. \tag{4.3}$$

In $d = 3+1$, we can evaluate the gauge field dependent quantity ζ to be

$$\zeta = \frac{1}{1+B_i^2}\left(P^0 + (E_i(1+B^2) - E\cdot BB_i)\Gamma_{0i} - \frac{1}{2}\epsilon^{ijk}B_k\Gamma_{ij} + E\cdot B\Gamma_{0123}\right). \tag{4.4}$$

Inserting this in the expression for J_2 we find

$$J_2 = 2\left(\Gamma_0 - \frac{1}{2}F_{ij}\Gamma_{0ij}\right)\lambda\,. \tag{4.5}$$

We can then use the Dirac brackets

$$\begin{aligned}\Big\{-P^0\Gamma_0\lambda + (E^i+P^i)\Gamma_i\lambda, \overline{\lambda}\Big\} &= \frac{1}{2},\\ \Big\{-P^0\Gamma_0\lambda + (E^i+P^i)\Gamma_i\lambda, A_j\Big\} &= \frac{1}{2}\Big\{-P^0\Gamma_0 + (E^i+P^i)\Gamma_i, A_j\Big\}\,\lambda\end{aligned} \tag{4.6}$$

to calculate the superalgebra

$$\Big\{\overline{Q}_a, Q_b\Big\} = \begin{pmatrix} -2(p^0\Gamma_0 + (p+e)^i\Gamma_i) & 2\Gamma_0 + f_{ij}\Gamma_{0ij} \\ 2\Gamma_0 - f_{ij}\Gamma_{0ij} & -2(p^0\Gamma_0 + 2(p-e)^i\Gamma_i)\end{pmatrix}, \tag{4.7}$$

(a, b take values $1, 2$). Recall that the lower case letters indicate the zero-modes of the fields; here they appear, since the charges Q are the integrals of the densities J. If one takes the limit for infinite rank, $n \to \infty$, one would recover the Yang-Mills superalgebra (with a redefinition of $Q_{1,2}$). In this sense, we can view the Born-Infeld theory as the generalisation of Yang-Mills M(atrix) theory to finite rank n, where the light cone direction is taken to be finite as well.

The superalgebra (4.7) is closely related to the algebra of 11-dimensional supergravity compactified on a four-torus. Concretely, if we define the 32 component supercharge

$$\mathcal{Q} = \begin{pmatrix}\Gamma_0 Q_1 \\ Q_2\end{pmatrix}, \tag{4.8}$$

and eleven-dimensional gamma-matrices

$$\gamma_i = \Gamma_i \otimes \sigma_1\,, \qquad \gamma_{11} = 1 \otimes \sigma_3\,, \tag{4.9}$$

the algebra becomes

$$\{\overline{\mathcal{Q}}, \mathcal{Q}\} = -2\left(p^0\gamma_0 + e^i\gamma_i + n\gamma_{11} + p^i\gamma_{i11} + \frac{1}{2}f_{ij}\gamma_{ij}\right)\,, \tag{4.10}$$

which is precisely the eleven-dimensional supersymmetry algebra (compactified on a four-torus), with e_i and n the compact momenta, and p_i, f_{ij} the components of the central charge (in the compact directions). (The central charge associated to the five-brane cannot take a finite value on a four-torus). This algebra is well known to enjoy the U-duality symmetry, SL(5) in the case of a four-torus compactification. Hence we can conclude that the BPS-mass deduced from (4.7) is invariant under SL(5). To compute it, we have to demand the matrix (4.7) to have eigenvalues equal to zero; this gives an equation for p_0. For a simple case, say $p_1, n \neq 0$ (and the rest of the quantum numbers zero), we find

$$p_0^2 = (n + p_1)^2\,. \tag{4.11}$$

By acting with $\mathrm{SL}(5,\mathbb{Z})$ we can transform to the case of general quantum numbers, to find

$$p_0^2 = n^2 + p_i^2 + e_i^2 + b_i^2 + 2\sqrt{(np_i - f_{ij}e_j)^2 + (p_ie_i)^2 + (p_ib_i)^2}\,. \tag{4.12}$$

Recall that strictly speaking we have only demonstrated this for configurations having $n = 1$. This expression for the energy of a BPS state coincides with the one found via the Bogomolny argument in the bosonic theory (equation (2.10)), where we put the moduli matrix $\mathcal{G}$ equal to the identity matrix.

5. BPS-equations

So far we found the U-invariant BPS mass spectrum from the supersymmetric Born-Infeld action. In this section we will go on to compute the BPS equations that configurations saturating this bound have to satisfy. They will of course ultimately turn out to be equivalent to those found in section 2. The strategy to find the equations is to demand a purely bosonic state to have the variations of the fermions equal to zero, for supersymmetry parameter ϵ satisfying

$$\left\{\overline{Q}, Q\right\}\epsilon = 0\,. \tag{5.1}$$

Looking at equations (4.2) this gives (to linear order in the fermions) the equation for ζ:

$$\bar{\epsilon}_1 + \bar{\epsilon}_2\zeta = 0\,, \tag{5.2}$$

or equivalently,

$$\mathcal{Z}\epsilon = \begin{pmatrix} 1 & \zeta \\ \overline{\zeta} & 1 \end{pmatrix} \epsilon = 0\,. \tag{5.3}$$

Here $\overline{\zeta} = -\Gamma_0\zeta^\dagger\Gamma_0$, and there is a relation $\overline{\zeta}\zeta = \zeta\overline{\zeta} = 1$ [12].

We have to insert the general form of ϵ satisfying (5.1) (given the central charges) in this equation, and then solve for ζ, i.e. determine the form of the gauge fields. For general quantum numbers, this seems to be rather complicated, until we notice that in the abelian case ($N = 1$) we may write

$$(N^2 + B_i^2)\mathcal{Z} = M^2\,, \tag{5.4}$$

with the matrix M defined as

$$M = \begin{pmatrix} -2(P^0\Gamma_0 + (P+E)^i\Gamma_i) & 2N\Gamma_0 + F_{ij}\Gamma_{0ij} \\ 2N\Gamma_0 - F_{ij}\Gamma_{0ij} & -2(P^0\Gamma_0 + (P-E)^i\Gamma_i) \end{pmatrix}. \tag{5.5}$$

Remarkably, this matrix is identical to the supersymmetry algebra (4.7), except that we have put in the complete fields for the zero-modes. We know how M transforms under the U-duality group, namely similarly as the supersymmetry algebra, which we will call m. (The transformation of M^2, and hence of ζ, on the other hand is rather complicated).

The BPS-equation is therefore found by demanding $M^2\epsilon = 0$ for an ϵ annihilated by the zero mode m of M; one can easily verify that this may be simplified to

$$M\epsilon = 0\,. \tag{5.6}$$

To extract the precise BPS condition, we write the condition $m\epsilon = 0$ in a convenient form as follows. Note that

$$\frac{1}{8}m\sigma_3 m\sigma_3 = |k|1\otimes 1 - (np_i - f_{ij}e_j)\Gamma_{0i}\otimes\sigma_1 - p\cdot b\Gamma_{0123}\otimes i\sigma_2 + p\cdot e1\otimes\sigma_3\,, \tag{5.7}$$

where k is the five-vector whose components are the coefficients of the other terms. This is precisely the same vector k that we encountered before. The expression is a projection operator (times a constant), it can be written as

$$\frac{1}{8}m\sigma_3 m\sigma_3 = |k| + k^i\tilde{\gamma}_i\,, \tag{5.8}$$

for suitable $\tilde{\gamma}_i$ satisfying a five dimensional Clifford algebra. From this we conclude that the condition $m\epsilon = 0$ can be expressed as

$$\epsilon = \sigma_3 m\sigma_3(|k| - k^i\tilde{\gamma}_i)\xi\,, \tag{5.9}$$

for arbitrary spinor ξ. Hence the BPS-equation becomes

$$M\sigma_3 m\sigma_3(|k| - k^i\tilde{\gamma}_i) = 0\,. \tag{5.10}$$

If we work this out, for general charges, this yields the set of conditions

$$\begin{aligned}
p_0(|k|M_{ij} + \epsilon_{ijklm}M_{kl}k_m) - P^0(|k|m_{ij} + \epsilon_{ijklm}m_{kl}k_m) &= 0 \\
M_{ij}(|k|m_{ij} + \epsilon_{ijklm}m_{kl}k_m) - P^0p^0|k| &= 0 \\
\epsilon_{ijklm}M_{jk}(|k|m_{lm} + \epsilon_{lmabc}m_{ab}k_c) - P^0p^0k^i &= 0 \\
k_iM_{ij} &= 0\,.
\end{aligned} \tag{5.11}$$

We naturally identified the matrix M acting on spinors as (5.5) with the five by five antisymmetric matrix M_{ij}, and likewise for the zero mode m_{ij}. Apart from the matrix M and its zero modes m, there also appears the hamiltonian, $(P^0)^2 = \frac{1}{2}M_{ij}M_{ij}$, and its zero mode $(p^0)^2 = \frac{1}{2}m_{ij}m_{ij} + 2|k|$. It can easily be checked that the last three equations all follow from the first one (one also has to use that $K \equiv M \wedge M = 0$), so that ultimately the BPS-equation just becomes

$$p_0(|k|M_{ij} + \epsilon_{ijklm}M_{kl}k_m) - P^0(|k|m_{ij} + \epsilon_{ijklm}m_{kl}k_m) = 0\,. \tag{5.12}$$

This is precisely the statement (2.11) obtained from the bosonic theory. Here, however, we immediately also get the factor of proportionality, $(P^0 - p_0)/p_0$.

To recapitulate, we have reproduced the BPS-bound and the BPS-equation from the supersymmetric theory. Now we would like to study solutions to this equation and interpret the meaning of space-dependent M_{45}, the rank of the gauge group. We will deal with this by noting that the definition of M from the relevant quantity $\mathcal{Z}$ leaves room for an extra rescaling by a fluctuating factor, which we will use to put M_{45} to a constant. Then we can use SL(5) as a solution generating transformation in the classical theory. After that we will study the quantum properties, by imposing the BPS equation on operators, and try to determine degeneracies of states of given quantum numbers m_{ij}.

6. Classical solutions of the BPS equations

We will now study solutions to the Born-Infeld BPS equations. Ideally, one would like to solve the BPS equations for a simple configuration, and then invoke the duality to generate the solutions at different quantum numbers. This will be indeed the strategy we will follow. There are two questions that need to be resolved for this to work, however. Firstly, in fact we only know the Born-Infeld theory for the abelian case. We will make the assumption that the non abelian generalisation exists and shares the features of the Yang-Mills theory in its BPS space discussed in section (3), that the fields effectively diagonalise and can be described as abelian fields on a longer circle. The second problem is that a duality transformation acting on the matrix M will in general not keep the component related to the rank a constant number. To solve this we will argue that M is not quite the gauge theory quantity, but is only related to it up to a local scale transformation, which may be used to 'gauge' the rank to a constant.

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To start with we will solve the BPS equation (5.12) in the simplest possible case, where the only non zero quantum numbers are n and p_1; we will also first take $n = 1$ and postpone the discussion on the non abelian case.

We have, with $p^0 = n + p$,

$$P^0 = N + P^1\,, \qquad E^1 = B^1 = 0\,, \qquad E^i = F^{1i}\,, \qquad P^i = 0\,, \tag{6.1}$$

where $i = 2, 3$. Since, in this particular case, $E^i = F^{0i}$, we recover the same equations as those valid in the Yang-Mills situation. As was done there, we choose the gauge $A_0 = A_1 = 0$, so that the gauge fields $A_{2,3}$ satisfy

$$(\partial_0 - \partial_1)A_i = 0\,. \tag{6.2}$$

To obtain a genuine gauge theory we of course set $N = n$. The solutions to the BPS equations are then parametrised by two functions $E_2(t + x_1)$ and $E_3(t + x_1)$, satisfying

$$\int E_{2,3} = 0\,, \qquad \int E_2^2 + E_3^2 = np_1\,. \tag{6.3}$$

Note that in principle the BPS equation does not rule out dependence on the coordinates x_2 and x_3; this would, however, not be consistent with the Maxwell equations (although often BPS equations imply equations of motion, this is not in general the case).

Now we will discuss the generalisation to the case that the rank $N > 1$. The problem is that the form of the action of the non abelian Born-Infeld theory is yet unknown, even in the bosonic case. Tseytlin proposed a definition of the theory using a symmetrised trace prescription [9], and it was argued in [10] that this should be the only action allowing a supersymmetric extension. The precise form of this extension has not been found. We will make some assumptions on the alterations that arise, guided by the situation in the Yang-Mills theory, and requiring that the duality group have some sensible action on the fields, at least in the BPS sector.

Some of the formulas appearing in this paper seem to have a natural generalisation to arbitrary N; in particular, one would expect the superalgebra to remain of the form presented in (4.7), with N as the coefficient of the $\Gamma_0 \otimes \sigma_1$ part, so that the relation to the eleven-dimensional algebra be preserved.

The introduction of a matrix M such as in the abelian case is now questionable, however. Merely inserting non-abelian matrices for the entries does not give the desired squaring to something of the form of $\mathcal{Z}$. Besides, it is not quite clear what we would mean by this. Though electric and magnetic fields are of course easily converted to Hermitian matrices, the prescription for the momenta P is already more complicated. One might for instance define it to be the anticommutator of the appropriate E's and F's. The matrix we would naturally introduce for the rank N is the $n \times n$ identity matrix, the trace of which of course is n. However, since duality mixes up the various components, these relations are no longer preserved. What is more, since N may acquire a different value, the size of the component matrices must change as well; it is hard to imagine a consistent action of the duality group on general fields with this property.

To resolve these problems we make the following assumptions. We assume that the BPS configurations of the non abelian theory will behave like those of the Yang-Mills theory, in the sense that the fluctuations of the fields will be simultaneously diagonalisable and that they will all depend again on only one coordinate. Again, there may be twists in the theory making the period of this coordinate n' times as long, where $n' = \gcd(n, e_1, b_1)$. This makes the theory effectively an abelian one. We further assume that this abelian BPS sector satisfies the generalisations to larger n of the equations presented in the original case $n = 1$. In this way, the SL(5) duality does have a reasonable action on the BPS sector of the theory.

Using these assumptions, the solution calculated above can be straightforwardly generalised to higher rank n. The equations are the same, so we again have fields that are left moving. The coordinate however now has periodicity n (or n' in the presence of the fluxes as argued above). The integral of the fields over this space then incorporates both the trace and the integration over the real domain. We choose the function N to be one, so that its integral indeed equals n. The other fields then again should have integrals as above.

So far we discussed only the BPS equations in the case of simple fluxes, with only p_1 and n unequal to zero. The equations were the same as those obtained in the Yang-Mills theory. We now turn to more general fluxes, where unlike the Yang-Mills case the BPS equations become more complicated. We will argue that their solutions can be found using SL(5) transformations.

$SL(5, \mathbb{Z})$ acts on the zero mode matrix m_{ij} by conjugation. Any configuration with non-zero $k = m \wedge m$ can be mapped to one with only non-zero n' and p'_1, such that $\gcd(k^i) = n' p'_1$ and $n' = \gcd(n, p_i, e_i, b_i)$ (see e.g. [17]). If the matrix M transforms similarly the BPS equations are also covariant under this group, provided one also transforms the five by five moduli matrix $\mathcal{G}$ (2.6) (off diagonal terms can also appear in this metric, and correspond to non-zero anti-symmetric tensor fields). An M_{ij} (with zero modes m_{ij}) solving the equation, will therefore, when transformed to M' (with zero modes m') also solve the BPS equation at the transformed moduli (note that P^0 is invariant if we also transform the moduli).

The new solution, however, will only correspond to a possible gauge theory solution if the new component M_{45} is a constant, to be identified with the rank. This is certainly the case under transformations belonging to $SL(2, \mathbb{Z}) \times SL(3, \mathbb{Z})$, which leave N invariant, but not in general.

The resolution is to recognise that the relation between the matrix M and the gauge theory fields E_i, B_i, P_i and N is not unambiguous. The original BPS equation was

$$\mathcal{Z}\epsilon = 0\,, \tag{6.4}$$

and we reformulated this in terms of a matrix M defined in equations (5.4,5.5). This reformulation is not unique however, as can be seen from the fact that ζ, the top righthand block of $\mathcal{Z}$, is a function only of the quotients M_{ij}/N:

$$\begin{aligned}\zeta &= \frac{1}{N^2+B^2}\Big[NP^0 + \Big(\frac{E_i}{N}(N^2+B^2) - \frac{E\cdot B}{N}B_i\Big)\Gamma_{0i} - \\ &\qquad - \frac{1}{2}\epsilon_{ijk}P^0 B_k\Gamma_{ij} + E\cdot B\Gamma_{0123}\Big] \\ &= \frac{1}{1+\left(\frac{B}{N}\right)^2}\Big[\frac{P^0}{N} + \Big(\frac{E_i}{N}\Big(1+\Big(\frac{B}{N}\Big)^2\Big) - (E\cdot B)\frac{B_i}{N^3}\Big)\Gamma_{0i} - \\ &\qquad - \frac{1}{2}\epsilon_{ijk}\frac{P^0 B_k}{N^2\Gamma_{ij}} + \frac{E\cdot B}{N^2\Gamma_{0123}}\Big]\,. \end{aligned} \tag{6.5}$$

Therefore, if we scale the matrix M, the physical fields present in ζ remain the same, demonstrating that M only equals the various fields up to a factor which may vary over spacetime. Given a solution for M, we may therefore derive the corresponding gauge fields by rescaling this M such that the entry in M_{45} becomes the appropriate constant, and then reading off the values of the fields.

The strategy to solve the BPS equations for general fluxes is then to first transform the fluxes by an element $g \in \mathrm{SL}(5,\mathbb{Z})$ to the simple case. Then we know the BPS equations; where formerly we put N to a constant, now we demand that the combination that under g^{-1} will be mapped to the 45 component of M is a constant. We further impose the equation

$$K = NP_1 - E_2B_3 + E_3B_2 = 0\,; \tag{6.6}$$

this leaves us with two arbitrary functions. On these we impose the condition that the zero modes of all fields are the correct ones. If we then transform back using g^{-1}, the resulting configuration will satisfy the BPS equation for these fluxes, will have the correct zero modes and will obey $N = n$. This guarantees that we have an allowed BPS configuration.

As mentioned earlier, fields satisfying the BPS equations and also the condition $M \wedge M = 0$ are not yet the complete story: we still have to demand them to satisfy the equations of motion and Bianchi identities. These will tell us how the fields depend on the coordinates. To show that in general the fluctuation parts of the E and B fields again are left movers, depending on the coordinate in the direction of the zero mode of the momentum of the fluctuating fields, we need the relation between F_{0i} and F_{ij} in the general case. The general expression for F_{0i} in terms of the electric and magnetic fields is

$$F_{0i} = \frac{(M^2)_{i5}}{P^0}\,. \tag{6.7}$$

We can now use the BPS equation (5.12) to derive

$$F_{0i} = \frac{1}{p^0} \left(m_{ij} + \epsilon_{ijklm} m^{kl} \hat{k}^m \right) M_{j5} \,. \tag{6.8}$$

We will analyse two situations, $k = k_1$ and $k = k_1 + k_4$. For other configurations the discussion is similar.

For k having only a component in the one-direction, we have that $M_{15} = 0$. Since the matrix multiplying M_{j5} is anti-symmetric, we see that F_{02} is a linear combination of B_3 and N. If we restrict to fluctuations, we therefore have

$$F'_{02} = \text{const} \cdot F'_{12} \,, \tag{6.9}$$

and similarly for F_{03}.

For k having also a component in the four-direction, we do not anymore have $B_1 = 0$; however, in this case B_1 is a constant, since from $k^i M_{ij} = 0$ we see that it is proportional to N. For the fluctuations, the relations therefore remain as above.

This demonstrates that in general we still require (fluctuating parts of) the fields A_2 and A_3 to satisfy the equations of a chiral boson, be it that the 'speed of light' now deviates from one.

This ends the discussion of the classical gauge field configurations in the BPS sector of the theory. In the next section we will again study the BPS sector, but there we will try to quantise the theory, with the BPS requirements as constraints.

7. Quantisation of the Born-Infeld BPS states

We have seen that $\mathrm{SL}(5, \mathbb{Z})$ is a symmetry of the BPS equations, and can be used to generate solutions. It is a different question whether the quantum theory will enjoy the symmetry, so that the duality group will be a symmetry of the space of states and the associated degeneracies.

The idea is not to quantise the complete Born-Infeld theory, but to consider only the phase space corresponding to configurations that are supersymmetric. This system is then to be quantised. The expectation is that, since supersymmetry is so robust under quantum corrections, this gives an adequate description of states and processes in the BPS sector.

To investigate this question we have to represent the fluctuating electric and magnetic fields as operators on a Fock space. The requirements will be that E and B satisfy the appropriate commutation relations, and that all the fields will have the prescribed integer expectation values when acting on states with the associated quantum numbers. We will construct such operators using the symmetry group and demonstrate that they have the desired properties.

In dealing with the rank N, we will use an analogue of the classical situation described before. There we saw that scaling all fields was a symmetry, so that N

could be put to the appropriate constant value. Here we will propose to identify this symmetry with a gauge symmetry, generated by the constraint $K = 0$. In the BPS sector, the system effectively reduces to a string theory, where K takes the role of the stress energy tensor. The choice N is constant is then simply the familiar light cone gauge.

We will start by quantising the simplest case, where only n and p_1 are unequal to zero. The BPS equations are

$$P_0 = N + P_1\,, \qquad E_1 = B_1 = 0\,, \qquad E_2 = B_3\,, \qquad E_3 = -B_2\,, \tag{7.1}$$

which in this case implies $F_{0i} = F_{1i}$ for $i = 2, 3$. The equations reduce the phase space to effectively only two independent functions, E_2 and E_3, (plus the six scalars), which can all be chosen to depend only on $t + x$. Canonical commutation relations imply:

$$[E_i(x_1), E_i(x_2)] = i\frac{1}{2}\partial^1\delta(x_1 - x_2)\,, \qquad \text{for } i = 2, 3. \tag{7.2}$$

As we saw, the fields E_i are functions of a coordinate whose periodicity is n times as long as that of x_1. The two fields can then be quantised as

$$E_2 = \sum_k \frac{1}{\sqrt{2n}}\alpha^2_{\frac{k}{n}} e^{i\frac{k}{n}(\tau+\sigma)}\,, \qquad E_3 = \sum_k \frac{1}{\sqrt{2n}}\alpha^3_{\frac{k}{n}} e^{i\frac{k}{n}(\tau+\sigma)}\,, \tag{7.3}$$

(and similarly for the scalars) where $[\alpha^i_{\frac{k}{n}}, \alpha^j_{\frac{l}{n}}] = \frac{k}{n}\delta^{i,j}_{k,-l}$, and σ runs from 0 to n. Then P_1 can be solved using the identity $NP_1 - (E \wedge B)_1 = 0$ (with N the identity matrix). Imposing that $\int P = p$ gives that the number of states is $d(np)$.

We propose to view the above computation as a string degeneracy computation in the light cone gauge in the following way. Let us assume an underlying theory where the components of M are all functions of space time (including the M_{45} component which represented the rank), and which satisfy commutation relations of the form

$$[M_{ij}(x), M_{kl}(y)] = i\epsilon_{ijklm}\partial^m\delta(x - y)\,. \tag{7.4}$$

Formally the index on the derivative runs over two additional dimensions, $4, 5$. We will, however, always demand that the BPS configuration only depends on time and the spatial coordinate in the direction of p (or rather κ, the first three components of k as introduced in section 3); all other derivatives are put to zero. Furthermore M satisfies the constraint $K = M \wedge M = 0$. The BPS conditions put various conditions on the components of M, and furthermore in the present case restrict all fields to depend only on $x_1 - t$. We label the remaining independent fields $N = \partial X^+$, $P_1 = \partial X^-, E_{2,3} = \frac{1}{\sqrt{2}}\partial X^{2,3}$, and the other six scalars $\frac{1}{\sqrt{2}}\partial X^i$. In this form the commutation relations (7.4) are precisely those of the left moving sector of a string, while the constraint K, which equals $K_1 = \partial X^+\partial X^- - \frac{1}{2}\partial X^i\partial X^i$, takes the form of the associated stress energy tensor. We now wish to interpret

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the choice $N = \partial X^+ = n$ as a gauge fixing of the symmetry generated by the local constraint K, analogous to the conventional light cone gauge. After this fixing, $P_1 = \partial X^-$ is to be solved from the constraint, and we end up with precisely the same representation of the various fields, with the correct commutation relations, as above. We therefore can view this description in terms of left moving bosons plus a gauge symmetry as effectively providing a way of generating a representation of the fields E_i, P_1, with the correct commutation relations and zero modes. Furthermore, the identification of K^1 with the stress energy tensor T explains the degeneracy formula, since we have a left moving string theory at level $L_0 = p^+p^- - \frac{1}{2}(p^i)^2 = np_1$.

It turns out that the generalisation to a theory with more degrees of freedom plus a gauge symmetry that we just introduced can be used also to quantise the theory for general quantum numbers. The strategy is to start with the previous case and, before fixing the gauge, acting on the fields with $\mathrm{SL}(5,\mathbb{Z})$. This will transform M to a new M where all components are linear combinations of the various ∂X. In particular M_{45} is now changed from ∂X^+ to a linear combination involving other ∂X^i as well. Using the fact that the theory with ten bosons and supersymmetry is Lorentz invariant, we may equivalently fix the new N to be equal to its zero mode n, and again solve for P_i; if we identify the gauge theory fields with the appropriate ∂X's *in the gauge where N is constant*, these fields will again satisfy the appropriate commutation relations, and have the right zero-modes. Furthermore, the degeneracy is again computed in the conformal field theory at level $|k|$.

To verify these assertions we will need to consider various subgroups of the duality group. First of all, the groups $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{SL}(3,\mathbb{Z})$, electro-magnetic duality and the symmetries of the torus, are already evident symmetries in the gauge theory description, without the introduction of the bosons, since they do not affect the rank N.

We will first consider the subgroup $\mathrm{SL}(4,\mathbb{Z})$ that leaves $k = k_1$ invariant. Inserting in the matrix M for the case of non zero n and p_1 the various ∂X's found above, by acting with the transformation we obtain a new M' which satisfies the new BPS equations. Since solutions of the resulting BPS equations still depend only on t and x_1 in some fixed combination, acting with the transformation leaves the commutation relations (7.4) invariant. After the transformation, the zero modes of M' are precisely the SL(4) transformed ones. In the previous case, we then picked a gauge where X^+ was proportional to the worldsheet time coordinate, so that $N = \partial X^+$ was a constant. Using Lorentz covariance, however, we may alternatively gauge fix any other combination of the X instead of X^+, without affecting the theory. Obviously we now pick the gauge where $N' = n'$. We may then again solve for P_1 ($P_{2,3}$ remain zero), and obtain a representation for the fields E and B satisfying the commutation relations. Since K^1 remains invariant, the stress energy tensor remains the same, as well as the level np_1, so that we find the same degeneracy.

The case where k is not left fixed is slightly more complicated; we will consider an SL(2) transformation on the simple case, changing $k_1 = np_1$ to $k'_1 = rk_1, k'_4 = sk_1$, with r, s mutually prime. Before application of this SL(2) transformation we denote $N = \partial X^+, P_1 = \partial X^-$ and $E_{2,3} = \partial X^{2,3}$ as before, and furthermore $B_2 = -E_3, B_3 = E_2$. Acting on these fields with the element of SL(2) results in a new configuration where N, E_2 and E_3 are r times their former expression, whereas $P_1, B_{2,3}$ remain unaffected. Furthermore, P_2, P_3 and B_1 also get a value different from zero.

If we would quantise the simple theory as before, and apply the transformation, we do of course get the desired degeneracy, $d(np_1)$. However, this is not quite correct, since the new configuration does not satisfy the required commutation relations: the new E and B commutator is r times too large. This can also be understood from the transformation of the M-commutation relations, as given in equation (7.4): formally, the only derivative that was non-zero, ∂^1, is transformed into $r\partial^1 + s\partial^4$. The new configuration can be taken to depend only on x_1, and we demand ∂^4 to be zero. This leaves us with a commutation relation that is r times its previous value.

To ensure the correct commutation relations after the SL(2) transformation, we therefore have to scale the commutation relations of the ∂X^i down by a factor r. However, if we then compute the degeneracy in the pre-transformation, simple state with only p_1 and n non-zero, with these rescaled relations we seem to get a degeneracy which is $d(rnp_1)$ rather than $d(np_1)$.

The situation is saved by the observation of [6], that the allowed twists generating long strings take values only in $S_{N'}$, rather than S_N, where $N' = \gcd(n, b_1)$. The fields after the transformation have a period which is r times smaller than naively expected. Therefore, not all oscillators are allowed: the oscillator numbers are r times larger than without this requirement. In the end, this combination of rescaling the commutation relations, plus rescaling the allowed periodicity of the fields, gives us again the correct result for the degeneracy: $d(\gcd(k)) = d(p \cdot \gcd(n, b_1))$.

Acting then on this configuration with SL(4) keeping the vector k invariant works similarly as in the case where k was pointing only in the 1-direction, and gives the same degeneracy. Other configurations of k can be trivially obtained using SL(2) $\times$ SL(3) transformations.

This concludes the demonstration of U-invariance of the degeneracies in the long string sector. Next we will investigate the contributions of multiple string states.

7.1 Multi-particle BPS states

So far we considered the degeneracy associated to the sector involving 'long strings', i.e. the configurations containing only one single BPS bound state. It is in principle also possible to construct multi-particle BPS states, whose charges add up to the total charge. In the symmetric product string sigma model these are identified with the states in those twisted sectors in the conjugacy class where N is partitioned in more than one piece. For these states to be BPS, it is necessary that all component states are annihilated by the same supersymmetry, or, equivalently, that the total energy is precisely the sum of the constituent BPS energies. In M-theory, this implies that the charge matrices of the component states should be proportional. In Yang-Mills theory this turned out not to be the case [6]; there was a greater freedom in the distribution of the charges over the constituents, in particular with respect to the value of the momentum zero mode. This was related to the fact that the Yang-Mills BPS mass was not symmetric under duality. In the present case, for generic moduli it is clear that all charges should align for the component masses to add up to precisely the composite BPS mass. This is of course also evident from the relation the supersymmetry parameters have to obey, $m\epsilon = 0$. We can therefore conclude that in the Born-Infeld case the counting of degeneracies, including those related to composite BPS states, gives the result expected from M-theory.

To give an explicit expression for the total U-invariant degeneracy, we first note that, since every component charge matrix m^α_{ij} is proportional to the total charge $m_{ij} = \sum_\alpha m^\alpha_{ij}$, the different compositions of m_{ij} in sets of m^α coincide with all possible ways one can partition the number $|m| \equiv \gcd(m_{ij})$ in a collection of n_ℓ states of $|m^\alpha| = \ell$, so that $\sum \ell n_\ell = |m|$. (The long string has $n_{|m|} = 1$, the other $n_\ell = 0$.) Each short, component, string has its own vector $k^\alpha = m^\alpha \wedge m^\alpha$, with $|k^\alpha| \equiv \gcd((k^\alpha)^i)$. The degeneracy associated to such a short string of length ℓ is therefore given by $d(|k^\alpha|)$. For the long string, we have $|k| = r|m|^2$, since the vector k is quadratic in the components of m. The number r is a fixed integer depending on the state. Therefore, we have $|k^\alpha| = r\ell^2$, so that a string of length ℓ can be in $1/2 \cdot d(r\ell^2)$ different bosonic states, and equally many fermionic ones. The total BPS-state is a symmetric combination of the partial ones, resulting in a degeneracy formula

$$\sum_{|m|} D(r,|m|) q^{|m|} = \prod_\ell \left(\frac{1+q^\ell}{1-q^\ell} \right)^{\frac{1}{2}d(r\ell^2)} , \tag{7.5}$$

where as mentioned $r = |k|/|m|^2$. The total degeneracy therefore depends on the two U-invariants, $|k|$ and $|m|$.

8. Discussion

We analysed the Born-Infeld theory of a compactified D3-brane, and argued for a realisation of $\mathrm{SL}(5,\mathbb{Z})$ U-duality on the 1/4 BPS sector of the theory (making a number of assumptions on the structure of the non-abelian theory). The action on the quantum numbers was well known already, the surprising fact was that there seems to exist an action of the duality also at the level of fields. These local degrees of freedom, assembled in the matrix $M_{ij}(x)$, are not exactly the fields of the gauge theory. Rather, M contains one more degree of freedom, which is compensated by a local symmetry acting on M. The Born-Infeld theory, in its BPS sector, was argued

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to be a gauge fixed version of this extended model: the gauge condition is that the rank is a constant. On the extended model SL(5) has a natural action; however, this action does not in general preserve the gauge. The action at the level of the Born-Infeld theory therefore has to be supplemented by a compensating transformation restoring the gauge.

Furthermore, by interpreting the model (reduced to the BPS sector) as a string theory in light cone gauge, we computed the degeneracies of the BPS states, and found them to be in agreement with the predictions of string/M theory.

We conclude by making some speculative remarks on some striking similarities of the model presented here to the self-dual tensor theory, the low energy theory living on the M theory five-brane. We will consider this theory compactified on a five-torus. In a hamiltonian approach [18], one can describe the theory in terms of a spatial two-form M_{ij} defined in terms of the self-dual field strength $F_{\mu\nu\rho}$ as

$$M_{ij} = \frac{1}{6}\epsilon_{ijklm}F^{klm}\,. \tag{8.1}$$

M_{ij} of course depends on all six spacetime coordinates. The energy and momentum densities of the theory are

$$P^0 = M_{ij}^2, \quad P^i = *(M\wedge M)\,, \tag{8.2}$$

which we recognise as our hamiltonian and K-vector, respectively. The Dirac brackets were calculated in [18] to be

$$[M_{ij}(x), M_{kl}(y)] = \epsilon_{ijklm}\partial^m\delta(x-y)\,. \tag{8.3}$$

This is of course the same as equation (7.4), but in this case all coordinates are truly five-dimensional. The SL(5) symmetry is evidently the purely geometrical symmetry of the five-torus.

A supersymmetric version of the M5 brane (in the PST formulation [20]) was studied in [19]. The supersymmetry algebra was calculated, and, like in our Born-Infeld calculation, it was shown to be precisely the M theory algebra. A compactified five-brane (on a five torus) of course has more charges; apart from the ten fluxes of M_{ij}, one generically also has the five momenta P^i, and the number of five-branes N. According to the correspondence between the momenta and our vector K, it seems that to make the connection to the three-brane theory one should demand all momenta to vanish. If we also set the five-brane number N to zero, the supersymmetry algebra becomes similar to the one of interest in this paper; furthermore, one finds the same BPS equation for 1/4 BPS states!

It is tempting to interpret the condition that the momenta vanish as the analogue of $L_0 - \overline{L}_0 = 0$ in string theory, and view the five-brane theory as an underlying theory for the gauge theory. Interestingly, the rank of the gauge theory would then be just one of the fluxes of the compactified five-brane.

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